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# Large amplitude ion-acoustic solitary waves in a relativistic multi-component plasma

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**Abstract** : Exact Sagdeev potential for a multidimensional relativistic multi-component plasma consisting of both positive and negative ions is derived without neglecting electron inertia. It is shown how the negative ion concentration, solitary $L.v_a$ - all play significant role in determining the nature of solitary waves

**Keywords** : Ion-acoustic solitary waves, large amplitude, relativistic multi-component plasma.

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## 1. Introduction

When the speed of the particles is comparable to the velocity of light, the relativistic effects play a significant role in the formation and propagation of solitary waves. For example, very high speed ions are observed in the solar atmosphere and interplanetary space. In the plasma sheet boundary of the Earth's atmosphere and in the Van Allen's radiation belts, high energy ion beams are frequently observed [1,2]. Propagation of ion-acoustic solitary waves has been studied both theoretically [3–6] and experimentally [7–9] by several authors. Most of the theoretical methods, however, used reductive perturbative technique (RPT) to derive KdV (Korteweg-deVries) or MKdV (Modified KdV) equation for nonlinear waves. But this technique is valid for small amplitude waves only [10]. Large amplitude solitary waves do exists in nature and in 1985, Nakamura [7] observed the large amplitude solitary waves in laboratory. Several approaches other than RPT are developed to study the nonlinear wave phenomena like Sagdeev's pseudopotential equation [11], Nonlinear Schrodinger equation, Sine Gordon equation [12] and Burger equation [13].

In 1983 Lonngren [14] studied solitary waves in multi-component plasma. Das [5] and Das and Tagare [6] also studied solitary waves in multi-component plasma with negative ions. This study is also extended to space plasmas through the derivation of Kadomtsev-Petviashvilli (K-P) equation [15] by Troven [16]. To study large amplitude solitary waves, Sagdeev's [10] pseudopotential approach is very useful particularly for travelling wave solutions. Originally, Sagdeev's pseudopotential was derived [17–19] mostly for unidirectional soliton propagation in plasma. Only recently, Roychoudhury *et al* [20] derived the multidimensional Sagdeev potential equation in multi-component plasma. But in their study, they considered plasma to be non-relativistic and also they neglected the mass of the electron. In this paper, our aim is to derive the generalized multidimensional Sagdeev equation in multi-component plasma taking into account the relativistic effect of the ions and taking into account the electron inertia. Here, we shall show how the negative ion concentration restrict the region of soliton solution. The effect of relativistic parameter in determining solitary wave solution is also discussed.

The organization of the paper is as follows. In Section 2 basic equations are derived. Sagdeev's pseudopotential is derived in Section 3. The conditions for existence of solitary waves are discussed in Section 4 and Section 5 is devoted to discussion and conclusions.

## 2. Basic equations

Our analysis is based on the continuity and momentum fluid equations for the ions and electrons and Poisson's equation which are given below :

For ions :

$$\frac{\partial n_\alpha}{\partial t} + \nabla.(n_\alpha v_\alpha) = 0, \tag{1}$$

$$\frac{\partial v_\alpha \gamma}{\partial t} + (v_\alpha . \nabla)(v_\alpha \gamma) = -q_\alpha \mu_\alpha \nabla \phi, \tag{2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v_\alpha^2}{c^2}}}$$

$$v_\alpha^2 = v_{\alpha x}^2 + v_{\alpha y}^2 + v_{\alpha z}^2.$$

$$v_\alpha . \nabla = v_{\alpha x}\frac{\partial}{\partial x} + v_{\alpha y}\frac{\partial}{\partial y} + v_{\alpha z}\frac{\partial}{\partial z}$$

For electrons :

$$\frac{\partial n_e}{\partial t} + \nabla.(n_e v_e) = 0, \tag{3}$$

$$Q\left(\frac{\partial}{\partial t} + v_e.\nabla\right)\nabla(v_e \gamma_e) = \frac{\partial \phi}{\partial x} - \frac{1}{n_e}\nabla . n_e \tag{4}$$

Poisson's equation :

$$\nabla^2 \phi = n_e - \sum q_\alpha n_\alpha, \tag{5}$$

where $Q = \frac{m_e}{m_\alpha}$, $m_e$ and $m_\alpha$ are electron and positive ion masses respectively. $\mu = \frac{m_i}{m_i}, \alpha = i, j$, represents positive and negative ions, $v_\alpha$ is the normalized velocity of a particle normalized to the ion-acoustic speed $c_s = \sqrt{\frac{kT_e}{m_i}}$. $q_\alpha$ is the charge ratio (+1 or −1 ). $n_e$, $n_\alpha$ are respectively the electron and ion densities. Space and time are normalized to the Debye length $\lambda_D = \sqrt{\frac{kT_e}{4\pi n_0 e^2}}$ and $\omega^{-1}$ respectively when the ion plasma frequency $\omega_i = \sqrt{\frac{4\pi n_0 e^2}{m_i}}$. $T_e$ is the electron temperature.

## 3. Pseudopotential analysis

In order to investigate the properties of the solitary wave solution of equations (1)–(5), we introduce a linear transformation as

$$\begin{aligned} \eta &= \xi[(l,m,n)(x,y,z) - Vt] \\ &= \xi[L.X - Vt] \\ &= \xi(lx + my + nz - Vt). \end{aligned}$$

Hence, we have

$$\begin{aligned} \frac{\partial}{\partial t} &= -V\xi\frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial x} &= l\xi\frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial y} &= m\xi\frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial z} &= n\xi\frac{\partial}{\partial \eta}. \end{aligned}$$

Eq. (1) thus becomes

$$\xi\left| -V\frac{\partial n_\alpha}{\partial \eta} + l\frac{\partial n_\alpha v_{\alpha x}}{\partial \eta} + m\frac{\partial n_\alpha v_{\alpha y}}{\partial \eta} + n\frac{\partial n_\alpha v_{\alpha z}}{\partial \eta} \right| = 0, \tag{6}$$

or

$$-V\frac{\partial n_\alpha}{\partial \eta} + \frac{\partial n_\alpha v_\alpha}{\partial \eta} = 0. \tag{7}$$

Integrating, we get

$$-Vn_\alpha + n_\alpha(L.v_\alpha) = c_1. \tag{8}$$

The initial conditions are $|n| \to 0, n_\alpha \to n_{\alpha 0}, v_\alpha \to v_{\alpha 0}$.

So $c_1 = -Vn_{\alpha 0} + n_{\alpha 0}[L.v_{\alpha 0}]$. And so

$$n_\alpha = n_{\alpha 0}\left[\frac{V - L.v_{\alpha 0}}{V - L.v_\alpha}\right]. \tag{9}$$

Hence, eq. (2) becomes

$$-V\frac{d(v_{\alpha x}\gamma)}{d\eta}+(L.v_{\alpha})\frac{d(v_{\alpha x}\gamma)}{d\eta}=-\mu_{\alpha}q_{\alpha}l\frac{d\phi}{d\eta}, \quad (10)$$

$$-V\frac{d(v_{\alpha y}\gamma)}{d\eta}+(L.v_{\alpha})\frac{d(v_{\alpha y}\gamma)}{d\eta}=-\mu_{\alpha}q_{\alpha}m\frac{d\phi}{d\eta}, \quad (11)$$

$$-V\frac{d(v_{\alpha z}\gamma)}{d\eta}+(L.v_{\alpha})\frac{d(v_{\alpha z}\gamma)}{d\eta}=-\mu_{\alpha}q_{\alpha}n\frac{d\phi}{d\eta}. \quad (12)$$

Now multiplying the eqs. (10), (11) and (12) by $l$, $m$ and $n$ respectively and adding we get

$$-V\frac{d}{d\eta}[L.v_{\alpha}\gamma]+l^2 v_{\alpha x}\frac{d}{d\eta}(v_{\alpha x}\gamma)+m^2 v_{\alpha y}\frac{d}{d\eta}(v_{\alpha y}\gamma)+$$

$$n^2 v_{\alpha z}\frac{d}{d\eta}(v_{\alpha z}\gamma)+lm\left| v_{\alpha y}\frac{d}{d\eta}(v_{\alpha x}\gamma)+v_{\alpha x}\frac{d}{d\eta}(v_{\alpha y}\gamma)\right.$$

$$+mn\left[v_{\alpha y}\frac{d}{d\eta}(v_{\alpha z}\gamma)+v_{\alpha z}\frac{d}{d\eta}(v_{\alpha y}\gamma)\right]+$$

$$\ln\left[v_{\alpha z}\frac{d}{d\eta}(v_{\alpha x}\gamma)+v_{\alpha x}\frac{d}{d\eta}(v_{\alpha z}\gamma)\right]=\mu_{\alpha}q_{\alpha}L^2\frac{d\phi}{d\eta}, \quad (13)$$

or

$$-V\frac{d}{d\eta}[L.v_{\alpha}\gamma]+\frac{1}{2\gamma}\frac{d}{d\eta}[L.v_{\alpha}\gamma]^2=-\mu_{\alpha}q_{\alpha}L^2\frac{d\phi}{d\eta}. \quad (14)$$

Now considering the propagation of solitary wave in the direction of $v_\alpha$ (*i.e.* $(L.v_{\alpha})^2=v_{\alpha}{}^2$), we get

$$\frac{1}{2\gamma}\frac{d}{d\eta}[L.v_{\alpha}\gamma]^2=v_{\alpha}\left[\gamma+v_{\alpha}\frac{d\gamma}{dv_{\alpha}}\right]\frac{dv_{\alpha}}{d\eta}, \quad (15)$$

and

$$\frac{d\gamma}{dv_{\alpha}}=\frac{v_{\alpha}}{c^2}\left(1-\frac{v_{\alpha}{}^2}{c^2}\right)^{\frac{-3}{2}}. \quad (16)$$

Using (14) and (15) in (13), we get

$$-Vd(L.v_{\alpha}\gamma)+\frac{v_{\alpha}}{\left(1-\frac{v_{\alpha}{}^2}{c^2}\right)^{\frac{3}{2}}}dv_{\alpha}=-\mu_{\alpha}q_{\alpha}L^2 d\phi. \quad (17)$$

Integrating (15), we get

$$-V(L.v_{\alpha}\gamma)+c^2\gamma=-\mu_{\alpha}q_{\alpha}L^2\phi+C, \quad (18)$$

when $v_{\alpha}\to v_{\alpha 0},\gamma\to\gamma(0)\ \dfrac{v_{\alpha 0}}{\sqrt{1-\frac{v_{\alpha 0}{}^2}{c^2}}}$ as $\phi\to 0$.

Hence, we get

$$V[L.v_{\alpha 0}\gamma_0-L.v_{\alpha}\gamma]+c^2(\gamma-\gamma_0)=\mu_{\alpha}q_{\alpha}L^2\phi. \quad (19)$$

Similarly, from eq. (3), we obtained

$$n_e = n_{e0}\frac{V-L.v_{e0}}{V-L.v_e}, \quad (20)$$

where $v_{e0}$ is the initial velocity of the electron and $n_{e0}$ is the initial electron density.
Considering $L.v_{e0}=0$, eq. (20) reduces to

$$n_e=n_{e0}\frac{V}{V-L.v_e} \quad (21)$$

Similarly, from eq. (4), we have

$$-Vd(L.v_e)+d(L.v_e)^2=\frac{2}{Q}L^2 d\phi-\frac{2}{Q}L^2\frac{1}{n_e}dn_e. \quad (22)$$

Integrating and using the boundary conditions $n_e\to n_{e0}, v_e\to 0$, as $\phi\to 0$, we get

$$\phi=\frac{Q}{2L^2}\left[(L.v_e)^2-2V(L.v_e)\right]+\log\frac{V}{V-L.v_e}. \quad (23)$$

Let us find the standard pseudopotential $\psi$ in the form

$$\frac{d^2\phi}{d\eta^2}=-\frac{\partial\psi}{\partial\phi} \quad (24)$$

where

$$\psi(\phi)=\sum_{\alpha}q_{\alpha}\psi(\phi)+\psi_e(\phi), \quad (25)$$

when

$$\psi_{\alpha}(\phi)=\int n_{\alpha}d\phi, \quad (26)$$

1

$$\psi_e(\phi)=\int n_e d\phi. \quad (27)$$

Integrating the above integral and using the suitable

(9)

boundary conditions, one has

$$\psi_\alpha = \sum_\alpha \frac{n_{\alpha 0}}{\mu_\alpha}[V - L.v_{\alpha 0}](L.v_\alpha \gamma - L.v_{\alpha 0}\gamma(0)). \tag{28}$$

Similarly,

$$\psi_e = -Vn_{e0}\left[\frac{1}{V - L.v_e} - Q(L.v_e) - \frac{1}{V}\right]. \tag{29}$$

Considering $n_{e0} = 1$, we have

$$\psi_e = 1 - \frac{V}{V - L.v_e} + QV(L.v_e) \tag{30}$$

## 4. Solitary wave solution

Whether the solitary wave solution of eq. (24) exists or not, can be determined from the nature of the pseudopotential $\psi$. Considering the very simple case where ion-drift velocity, electron inertia, relativistic effect are negligible and also considering only one species of positive ion, we get

$$\psi_e = 1 - e^{\varphi}, \tag{31}$$

$$\psi_i = V^{\cdot}\ 1 - \left| 1 - \frac{\cdot}{V^2}\right. \tag{32}$$

Sagdeev shows that in this case, solitary wave solutions exist for 1 < V < 1.6.

For a solitary wave solution, the pseudoparticle starts at a position $\phi = 0$ with a small velocity $\frac{d\phi}{d\eta}$ and it will be reflected back at some positive $\phi = \phi_0$ and then come back to $\phi = 0$. Then the condition for the potential well is

$$\left.\frac{d^2\psi}{}\right|_{\phi=0} \langle 0 \tag{33}$$

Also $\psi(\phi)$ will be negative from $\phi = 0$ to $\phi = \phi_0$. The condition is that the physically complex $\psi$ will not be allowed because this implies complex ion density which is impossible. So, here the condition for the existence of soliton solution is

$$\frac{V - L.v_{\alpha 0}}{(V - L.v_\alpha)^3\gamma^3} - \frac{}{(V - L.v_e)[1 - Q(V - L.v_e)^2]} \langle 0 \tag{34}$$

and

$$\psi(\phi_0) = 0, \tag{35}$$

where $\phi_0$ is the point where $\psi(\phi)$ crosses the $\phi$ axis from below.

## 5. Discussion and conclusions

The exact pseudopotential $\psi(\phi)$ is derived for multicomponent multidimensional plasma. In Figure 1,

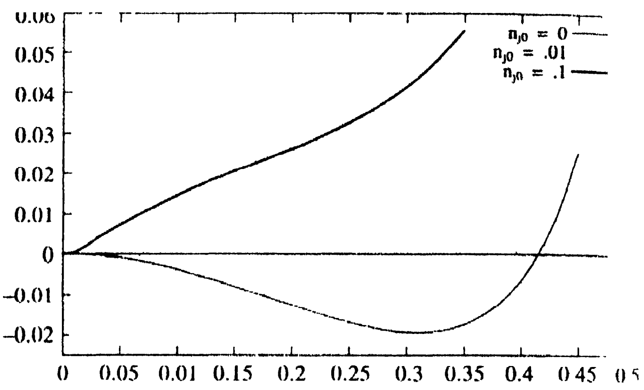


**Figure 1.** $\psi(\phi)$ is plotted against $\phi$ for different values of $n_{j0}$ = 0, 0.01, 0.1 Other parameters are $V = 31$, $L.v_{\alpha 0} = 30$, $\frac{L.v_{\alpha 0}}{} = 0.01$

$\psi(\phi)$ is ploted against $\phi$ for different values of negative ion concentration ($n_{j0}$ = 0, 0.01, 0.1). Other parameters are $V = 31$, $L.v_{\alpha 0} = 30$, $\frac{L.v_{\alpha 0}}{c} = 0.01$. For $n_{j0}$ = 0 or 0.01, $\psi(\phi)$ crosses the $\phi$ axis from below at a positive value of $\phi = \phi_m$ (say), where $\phi_m$ is the amplitude of the solitary wave. But for $n_{j0} = 0.1$, $\psi(\phi)$ is positive throughout the region and so no soliton solution would exist. Hence from this figure, it is seen that a small concentration of negative ion ($n_{j0}$) reduces the amplitude solitary waves but for a large concentration ($n_{j0} = 0.1$) of negative ions, no soliton solution would exist.

To see the effect of the relativistic parameter $\frac{L.v_{\alpha 0}}{c}$ in Figure 2, $\psi(\phi)$ is plotted against $\phi$ for $\frac{L.v_{\alpha 0}}{} = 0.001$, 0.01 and 0.1. Other parameters are $n_{j0} = 0.01$ and $V = 31.37$. It is seen from this figure that for small values of relativistic parameter $\left|\frac{L.v_{\alpha 0}}{} = 0.01 \text{ or } 0.001\right|$, the amplitude of solitary waves remains almost same but for

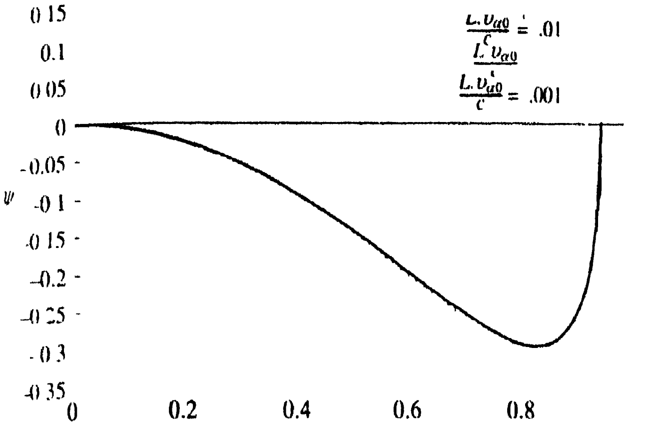


**Figure 2.** $\psi(\phi)$ is plotted against $\phi$ for different values of $\frac{L.v_{a0}}{c} = 0, 0.01, 0.1$ $n_{j0} = 0.01$ and $V = 31.37$. Other parameters are same as those in Figure 1.

a larger value of reltivistic paramcter $\left.\frac{L.v_{a0}}{c} = 0.1\right\}, \psi(\phi)$ becomes negative through out the region and hence no solution exists in this case.

To see the effect of solitary wave velocity $V$, $\psi(\phi)$ is plotted against $\phi$ for $V = 31$, $31.37$ and $31.6$ in Figure 3. Other parameters are $n_{j0} = 0.01$ and $\frac{L.v_{a0}}{c} = 0.01$. It is clearly seen from this figure that $\psi(\phi)$ does not cross the $\phi$ axis from below for $V > 31.37$. Hence, $V = 31.37$ is the critical value of the solitary wave velocity above which no soliton solution would exist. The shape of the solitary waves may easily be found out from the relation

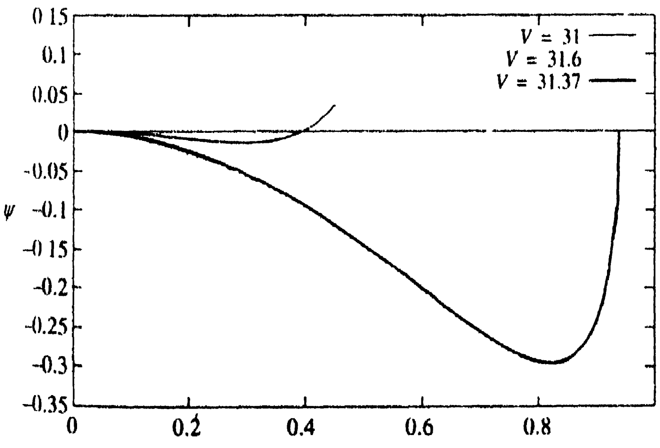


**Figure 3.** $\psi(\phi)$ is plotted against $\phi$ for different values of $V = 31$, $31.37$, $31.6$. and $n_{j0} = 0.01$. Other parameters are same as those in Figure 1.

$$\xi = \pm \int^{\phi} \frac{d\phi}{\sqrt{-2\psi(\phi)}}. \tag{36}$$

Hence, we can conclude that the concentration of negative ions has a significant role in the existence in multi-dimensional multicomponent plasma. The solitary wave velocity and the relativistic parameter have also significant roles in determining the region of existence as well as in the shape and the amplitude of the solitary waves.

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